HIGH ORDER ACCURACY METHODS FOR PDEs with SHOCKS and UNCERTAINTIES

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Abstract
The aim of this project is the construction of efficient high fidelity schemes for systems of PDEs that contain shock waves.

We advocate here the Discontinuous collocation method, i.e. multi-domain spectral method with stable and conservative penalty interface conditions. In this work, the previously developed methodology is generalized to inhomogeneous grid to simulate the two dimensional supersonic injector-cavity system. Non-physical modes generated at the domain interfaces due to the spatial grid inhomogeneity is minimized using the new weighted multi-domain spectral penalty method. The method yields accurate and stable solutions of the injector-cavity system which agree well with experiments qualitatively. Through the direct numerical simulation of the injector-cavity system using the weighted method, the geometric effect of the cavity wall on the pressure fluctuation is investigated. It is shown that the recessed slanted cavity attenuates pressure fluctuations inside cavity enabling the cavity to act potentially as a stable flameholder for scramjet engine.

Multi-domain spectral penalty method for injector-recessed cavity system

The major developments of the current work are
1. Stable and conservative penalty type interface conditions are derived for the inhomogeneous grid.
2. The weighted spectral penalty method is used to minimize the non-physical growth modes at the inhomogeneous domain interfaces.
3. The weighted spectral penalty method is applied successfully to the two dimensional reactive supersonic injector-recessed cavity flameholder.

A crucial requirement from DNS computations of the injector-cavity system is to resolve the hydrogen jet injector without causing any instability or nonphysical growing modes at the domain interfaces. We use a smaller subdomain with higher order polynomials to resolve the narrow jet. In Figure 1 the local domain configuration

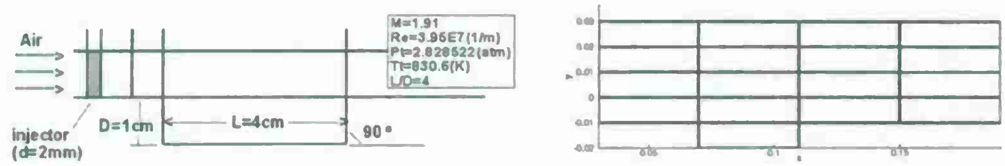

Figure 1: Left: Local domain configuration of the normal injector-cavity flame-holder. Right: Local domain configuration of the normal cavity without injector.

is given for the cavity flameholder with (left figure) and without (right figure) the injector. The ratio of the injector to the cavity length scale is about $O(10^{-1})$. The local domain configuration shown in the right figure is the typical domain system used for a homogeneous grid. The grid system in the left figure is inhomogeneous as the local injector is in the narrow domain. The stability analysis has been done with the assumption that each subdomain has the same length but can have different polynomial orders. With different polynomial orders, the stability is still maintained. We show that the stability can be also maintained with the different domain length.

Consider the following one-dimensional conservation laws:

$$\frac{\partial q(x,t)}{\partial t} + \frac{\partial f(q(x,t))}{\partial x} = 0, \quad x \in \mathbf{R}, \quad t > 0. \tag{1}$$

Here $q(x,t)$ is the state vector and $f(q(x,t))$ is the flux vector. Theorems for the multi-domain Legendre penalty method have been provided under the assumption that each subdomain has the same domain length but the polynomial orders of approximation can be different. The different polynomial order in each subdomain makes the grid system inhomogeneous.

For simplicity, we consider two subdomains $\Omega^I = [x_L, 0]$ and $\Omega^{II} = [0, x_R]$, for which the domain interface is at $x = 0$. Furthermore the left subdomain uses the polynomial order of N and the right subdomain of M and N is not necessarily the same as M. The Legendre multi-domain spectral penalty method is then given by

$$\begin{aligned}
\frac{\partial q_N^I}{\partial t} + \frac{\partial I_N^I f(q_N^I)}{\partial x} = \; & \tau_1 Q_N(x)[f^+(q_N^I(0,t)) - f^+(q_M^{II}(0,t))]+ \\
& \tau_2 Q_N(x)[f^-(q_N^I(0,t)) - f^-(q_M^{II}(0,t))]+ \quad \mathcal{B}(q_N^I(x_L,t)) + SV(q_N^I), \\
\frac{\partial q_M^{II}}{\partial t} + \frac{\partial I_M^{II} f(q_M^{II})}{\partial x} = \; & \tau_3 Q_M(x)[f^+(q_M^{II}(0,t)) - f^+(q_N^I(0,t))]+ \\
& \tau_4 Q_M(x)[f^-(q_M^{II}(0,t)) - f^-(q_N^I(0,t))]+ \quad \mathcal{B}(q_M^{II}(x_R,t)) + SV(q_M^{II}).
\end{aligned} \tag{2}$$

Here q_N^I denotes the numerical approximation of $q(x,t)$ in Legendre polynomial of order N in Ω^I and q_M^{II} of order M in Ω^{II}. $\mathcal{B}$ is the boundary operator at the end points, i.e. $x = x_L, x = x_R$ and SV is the spectral vanishing-viscosity terms. I_N^I

and I_M^{II} are the Legendre interpolation operators for the left and right subdomains respectively. Q_N and Q_M are the polynomials of order N and M respectively defined to vanish at the collocation points except at the boundary or interface points, that is, for Ω^I, $Q_N(x_i) = 0$ for $i = 1, \cdots, N-1$ and $Q_N(x_i) = 1$ for $i = 0, N$. The positive and negative fluxes f^+ and f^- are defined by

$$f^{\pm} = \int S\Lambda^{\pm}S^{-1}dq, \qquad A \equiv \frac{\partial f}{\partial q} = S\Lambda S^{-1}. \tag{3}$$

The Jacobian matrix A is assumed to be symmetric. Λ^+ and Λ^- are the diagonal matrices composed of positive and negative eigenvalues of A respectively such as $\Lambda = \Lambda^+ + \Lambda^-$. S and λ are the variables related to the characteristics and its direction of propagation. τ_1, τ_2, τ_3 and τ_4 are the penalty parameters and all are constants and the boundary terms are ignored. For the following discussion we define the discrete Legendre norm $(p,q)_N := \sum_{i=0}^{N} p(x_i)q(x_i)\omega_i$. x_i are the Legendre Gauss-Lobatto collocation points and $\omega_i = \frac{2}{N(N+1)[L_N(\xi(x_i))]^2}$ where ξ is the map from x to the Legendre Gauss-Lobatto points over $[-1,1]$. If $pq \in P_{2N-1}$, the discrete sum is exact, i.e. $(p,q)_N = \int_{-1}^{1} p(\xi(x))q(\xi(x))d\xi$. We define the weight vector $\vec{\omega}_N^I$ as the weight vector in Ω^I with $N+1$ components such as $\vec{\omega}_N^I = (\omega_0^I, \cdots, \omega_N^I)^T$. We note that ω_N^I without the vector symbol denotes the last component of $\vec{\omega}_N^I$. It can be shown that the scheme (3) is conservative and stable if

$$\frac{\Delta^I}{2}\tau_1\omega_N^I - \frac{\Delta^{II}}{2}\tau_3\omega_M^{II} = 1, \quad \frac{\Delta^I}{2}\tau_2\omega_N^I - \frac{\Delta^{II}}{2}\tau_4\omega_M^{II} = 1. \tag{4}$$

and

$$2\tau_1\omega_N^I \le \frac{2}{\Delta^I}, \quad 2\tau_3\omega_M^{II} \le -\frac{2}{\Delta^{II}}, \quad 2\tau_2\omega_N^I \ge \frac{2}{\Delta^I}, \quad 2\tau_4\omega_M^{II} \ge -\frac{2}{\Delta^{II}}, \tag{5}$$

$$(\tau_1\omega_N^I - \tau_3\omega_M^{II})^2 - 2(\tau_1\omega_N^I\frac{2}{\Delta^{II}} - \tau_3\omega_M^{II}\frac{2}{\Delta^I}) + \frac{2}{\Delta^I}\frac{2}{\Delta^{II}} \le 0,$$

$$(\tau_2\omega_N^I - \tau_4\omega_M^{II})^2 - 2(\tau_2\omega_N^I\frac{2}{\Delta^{II}} - \tau_4\omega_M^{II}\frac{2}{\Delta^I}) + \frac{2}{\Delta^I}\frac{2}{\Delta^{II}} \le 0, \tag{6}$$

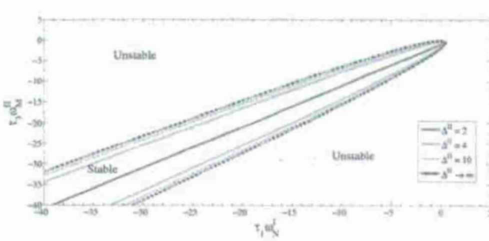

Figure 2: Stability regions. $\Delta^I = 2$ and $\Delta^{II} = 2, 4, 10, \infty$. The concaved curves separate the stable and unstable regions below and above the curves respectively for each given Δ^{II}.

Figure 2 shows the stability regions for $\tau_1\omega_N^I$ and $\tau_2\omega_M^{II}$ with various Δ^{II} for which $\Delta^I = 2$ is used. When $\Delta^{II} = 2$, that is, when the grid is homogeneous, the stability region is simply given as a linear line shown as the blue solid straight line in the figure. As the domain size ratio between Ω^I and Ω^{II} increases the stability region becomes broader. The green and red dotted lines in the figure represent the stability region for $\Delta^{II} = 4$ and $\Delta^{II} = 10$, respectively. The outermost black long dash-dot line represents the limit of the stability region, i.e. for $\Delta^{II} \to \infty$. The limit line is given by $(\tau_1\omega_N^I - \tau_3\omega_M^{II})^2 + 2\tau_3\omega_M^{II} \leq 0$ and is independent of the value of Δ^{II}. If $\tau_1 = \tau_4 = 0$, the penalty interface conditions are basically the same as the upwind methods. If $\tau_1 = \tau_2$ and $\tau_3 = \tau_4$ then the scheme does not split the flux into the positive and negative ones but uses the flux itself in the penalty terms.

The numerical simulation results of supersonic reactive cavity flow presented show that the upwind characteristic interface conditions are not enough to ensure the smooth solutions across the interfaces. By weighting the incoming fluxes against the outgoing fluxes, the nonphysical modes at the domain interfaces can be reduced. The weighted penalty method is based on the characteristic decomposition and it does not modify the stability conditions associated with $\mathbf{A}_\nu \cdot \mathbf{q}$ and $\mathbf{A}_\nu \cdot \partial\mathbf{q}$ for the Navier-Stokes equations. The weight, however, can not be arbitrarily large due to the CFL restriction.

Cavity has been actively used as a flame-holder in scramjet engine. The injector-cavity system is illustrated in Figure 1. The cavity system is categorized into 4 different types such as open, closed, transitional-closed and transitional-open depending on the length scale of cavity. The cavity system with the length-to-depth ratio $L/D < 7 \sim 10$ is called an open cavity as the upper shear layer reattaches itself at the back face. Under the shear layer formed over cavity, the flows with the hydrogen fuel are possibly captured inside cavity and generate the recirculation zone. Here we use the length-to-depth ratio $L/D = 4\text{cm}/1\text{cm} = 4$, that is, we use the open cavity. The generated recirculation interacts with the shear layer and the acoustic waves inside cavity. The radicals from the chemical reaction between the hydrogen and oxygen gases reside inside cavity and trigger the auto-ignition of the supersonic engine. In principle, the more stable and longer recirculation is maintained, the more efficient fuel performance can be achieved. Since the injection of the fuel in the combustor is necessary, the injection emerges as another important key parameter for the optimal configuration of the cavity flame-holders.

Figure 3 shows the effect of the grid inhomogeneity on the solution. The subdomain containing the injector has a smaller domain length than the channel subdomains. The left figure shows the solution based on the averaging method and the right shows the solution based on the weighted penalty method. The figures clearly show that the averaging interface condition yields a nonphysical concentration near the domain interface while the weighted penalty interface condition yields smooth solutions across the interfaces.

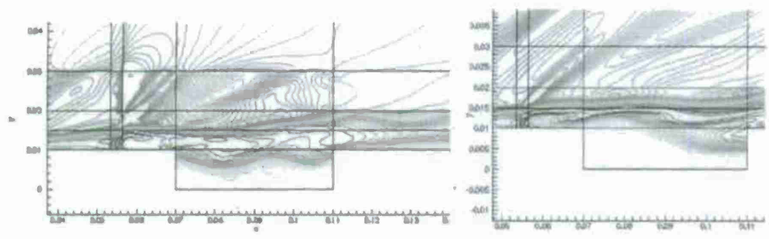

Figure 3: Density contour of injector-recessed cavity flow. The left and right figures show the solution using the averaging interface conditions and the weighted penalty interface conditions respectively.

For the numerical experiments, we consider two different injectors, the narrow and broad injectors in front of recessed cavities with 30 and 90 degrees aft wall. Figure 4 shows the water contours for both cases. By placing the injector ahead of the cavity front wall, the pressure fluctuations are reduced and the sharp gradients found near the corner of the aft wall are also weakened as the shear layer is being developed. The figures show that the broader injector has more enhanced shear layer growth over the cavity than the narrow jet. Both the broad and narrow injector systems also show that they have weaker flow gradients near the aft wall than the flow gradients obtained in our previous work without the injector.

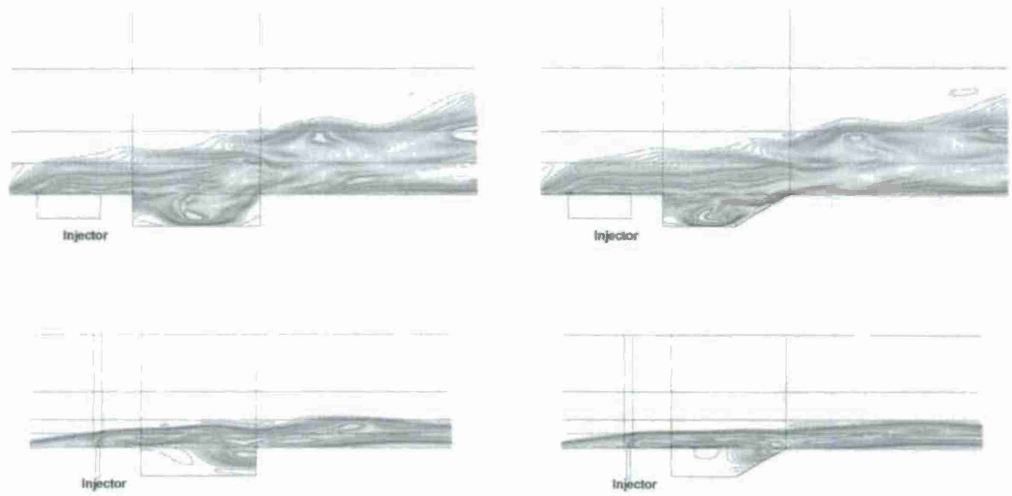

Figure 4: Water production and transport for 90° and 30° cavity walls: the upper figures shows the broad injector-cavity system and the bottom figures the water contours of the narrow injector-cavity system at $t = 0.225$ms, respectively.

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Publications

W. S. Don, D. Gottlieb & J. H. Jung, *A weighted multi-domain spectral penalty method with inhomogeneous grid for supersonic injective cavity flows* Communication in Computational Physics, 2008, under revision

R. Borges, M. Carmona, B. Costa & W. S. Don, *An improved weighted essentially non-oscillatory scheme for hyperbolic conservation laws*, Journal of Computational Physics **207** 6, 2008

G. Jacobs & W. S. Don, *High Order WENO-Z finite difference methods based Particle-Source-in-Cell method for Particle-Laden flows with Shock*, Journal of Computational Physics, 2008, submitted

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Honors & Awards Received

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